2023-24 MATH2048: Honours Linear Algebra II

Homework 10 Answer

Due: 2023-12-04 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^{\perp}$.

   Solution. Note that T is a projection $\iff T^2 = T$.
   If T is the orthogonal projection of V on W. Then $R(T) = W$, $R(T)^{\perp} = N(T)$ and $R(T) = N(T)^{\perp}$. Let $U = I - T$.

   - Claim 1: $R(U) = W^{\perp}$
     For any $v \in V$, $U(v) = v - T(v)$ and $T(U(v)) = T(v) - T^2(v) = 0$. So $U(v) \in N(T) = W^{\perp}$ i.e. $R(U) \subset W^{\perp}$.
     For any $x \in W^{\perp} = N(T)$ there exists $x \in V$ such that $U(x) = x - T(x) = x$. So $x \in R(U)$ i.e. $W^{\perp} \subset R(U)$.
   - Claim 2: $N(U) = W$
     For any $x \in W$, $U(x) = x - T(x) = x - x$. So $x \in N(U)$ i.e. $W \subset N(U)$.
     For any $x \in N(U)$, $U(x) = 0 \implies x = T(x) \in R(T) = W$. So $N(U) \subset W$.

   Therefore $R(U) = W^{\perp} = N(U)^{\perp}$ and $N(U) = W =^{\nabla} (W^{\perp})^{\perp} = R(U)^{\perp}$, where ∇ holds because W is finite-dimensional.

2. Let T be a linear operator on a finite-dimensional inner product space V.

   (a) If T is an orthogonal projection, prove that $\|T(x)\|^2 \le \|x\|^2$ for all $x \in V$. Give an example of a projection for which this inequality does not hold. What can be concluded about a projection for which the inequality is actually an equality for all $x \in V$?

(b) Suppose that T is a projection such that $\|T(x)\|^2 \leq \|x\|^2$ for $x \in V$. Prove that T is an orthogonal projection.

Solution.

(a) If T is a projection, the eigenvalues of T must be 0 or 1. Let $\beta_0 = \{v_1, ..., v_k\}$ be a basis for the eigenspace E_0 and $\beta_1 = \{v_{k+1}, ..., v_n\}$ be a basis for the eigenspace E_1.

Sicne T is an orthogonal projection, it's diagonalizable, which implies $V = E_1 \oplus E_0$. Let $\beta = \beta_0 \cup \beta_1$. Then β is an eigen basis for V. For any $x \in V$, there exists scalars a_i such that $x = \sum_{i=1}^n a_i v_i$.

Then $\|T(x)\|^2 = \|\sum_{i=k+1}^n a_i v_i\|^2 = \sum_{i=k+1}^n |a_i|^2 \leq \sum_{i=1}^n |a_i|^2 = \|\sum_{i=1}^n a_i v_i\|^2 = \|x\|^2$.

The example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $T((a,b)^t) = (a+b, 0)^t$. Since $T^2 = T$, T is a projection. But $\|T((1,1)^t)\|^2 = \|(1+1, 0)\|^2 = 4 > 2 = \|(1,1)^t\|^2$.

By the analysis above, the inequality is an equality for all $x \in V$ if and only if $k = 0$ if and only if $\dim(E_0) = 0$ if and only if T is invertible.

3. (a) Let A and B be commuting square matrices, i.e., $AB = BA$. Show that the binomial formula can be applied to $(A+B)^n$, i.e.,

$$(A+B)^n = \sum_{k=0}^n \binom{n}{k} A^{n-k} B^k,$$

where $\binom{n}{k}$ is the binomial coefficient.

(b) Let A the Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

Find A^4. (Hint: Use part (a).)

Solution.

(a) If $AB = BA$, then $A^k B^l = B^l A^k$ for any nonnegative integer k, l. Therefore, $(A+B)^n = \sum_{k=0}^n \binom{n}{k} A^{n-k} B^k$.

(b) Let $J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $J^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $J^k = O_{3\times 3}$ for any $k \geq 3$.

Note that λI and J commute. Then $A = \lambda I + J$ and $A^4 = (\lambda I + J)^4 = \lambda^4 I + 4\lambda^3 J + 6\lambda^2 J^2 + 4\lambda J^3 + J^4 = \lambda^4 I + 4\lambda^3 J + 6\lambda^2 J^2$. Threrfore $A^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}$.

4. Let V be the real vector space of functions spanned by the set of real valued functions $\{1, t, t^2, e^t, te^t\}$, and T the linear operator on V defined by $T(f) = f'$.

   Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

   Solution. Let $\beta = \{1, t, t^2, e^t, te^t\}$. Then $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

   The characteristic polynomial is $f_T(t) = \det([T]_\beta - tI) = t^3(t - 1^2)$.

   - For eigenvalue 0

     $[T]_\beta - 0I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$; $([T]_\beta - 0I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$;

     $([T]_\beta - 0I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$; $([T]_\beta - 0I)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

     Note that $\dim(N(([T]_\beta - 0I)^3)) = 3 = \mu_T(0)$. Find $x = (0, 0, 1, 0, 0)^t \in \mathbb{R}^5$ such that $([T]_\beta - 0I)^3 x = 0$ and $([T]_\beta - 0I)^2 x \neq 0$.
     Then $\{([T]_\beta - 0I)^2 x, ([T]_\beta - 0I)x, x\} = \{(2, 0, 0, 0, 0)^t, (0, 2, 0, 0, 0)^t, (0, 0, 1, 0, 0)^t\}$ is a basis for the generalized eigenspace $\tilde{K}_0$ of $L_{[T]_\beta}$.

   - For eigenvalue 1

$$[T]_\beta - 1I = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \ ([T]_\beta - 1I)^2 = \begin{pmatrix} 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$([T]_\beta - 1I)^3 = \begin{pmatrix} -1 & 3 & -6 & 0 & 0 \\ 0 & -1 & 6 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that $\dim(N(([T]_\beta - 1I)^2)) = 2 = \mu_T(1)$. Find $y = (0,0,0,0,1)^t \in \mathbb{R}^5$ such that $([T]_\beta - 1I)^2 y = 0$ and $([T]_\beta - 1I)y \neq 0$.

Then $\{([T]_\beta - 1I)y, y\} = \{(0,0,0,1,0)^t, (0,0,0,0,1)^t\}$ is a basis for the generalized eigenspace $\tilde{K}_1$ of $L_{[T]_\beta}$.

Then $\{2, 2t, t^2\}$ is a basis for the generalized eigenspace K_0 of T and $\{e^t, te^t\}$ is a basis for the generalized eigenspace K_1 of T.

Let $\gamma = \{2, 2t, t^2, e^t, te^t\}$, then the jordan form of T is $[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

5. Let $\gamma_1, \gamma_2, ..., \gamma_p$ be cycles of generalized eigenvectors of a linear operator T corresponding to an eigenvalue λ. Prove that if the initial eigenvectors are distinct, then the cycles are disjoint.

   Solution. Suppose the initial eigenvectors are distinct. If the cycles are not disjoint, then we have some element x in at least two cycles, say γ_1 and γ_2. Consider the smallest integer q such that $(T - \lambda I)^q(x) = 0$. We see that $(T - \lambda I)^{q-1}(x)$ is the initial eigenvector for both γ_1 and γ_2, which is a contradiction. Hence, the cycles must be distinct.